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# PRODUCTS OF GROUPS AND GROUP CLASSES

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ABSTRACT

Let $\mathcal{X}$ be a Schunck class, and let the finite group $G = AB = BC = AC$ be the product of two nilpotent subgroups $A$ and $B$ and an $\mathcal{X}$-subgroup $C$. If for every common prime divisor $p$ of the orders of $A$ and $B$ the cyclic group of order $p$ is an $\mathcal{X}$-group, then $G$ is an $\mathcal{X}$-group. This generalizes earlier results of O. Kegel and F. Peterson. Some related results for groups of the form $G = AB = AK = BK$, where $K$ is a nilpotent normal subgroup of $G$ and $A$ and $B$ are $\mathcal{X}$-groups for some saturated formation $\mathcal{X}$, are also proved.

## 1. Introduction

A well-known theorem of Kegel [11] and Wielandt [19] states the solubility of every finite group $G = AB$ which is the product of two nilpotent subgroups $A$ and $B$; see [4], Theorem 2.4.3. Amberg [1] and Pennington [15] have shown that the Fitting subgroup $F$ of such a product is factorized, i.e. $F = (A \cap F)(B \cap F)$ and $A \cap B \subseteq F$; see [4], Lemma 2.5.7. If there is a third subgroup $C$ such that $G = AB = AC = BC$, Kegel [12] and Pennington [14] proved that $G$ is nilpotent (supersoluble) whenever $C$ is nilpotent (supersoluble). Moreover, Peterson [16] has shown that if $\mathfrak{F}$ is a saturated formation containing all finite nilpotent groups, then $G$ is an $\mathfrak{F}$-group whenever $C$ is an $\mathfrak{F}$-group; see [4], Theorem 2.5.10.

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Our first theorem extends these results on trifactorized groups to Schunck classes. Recall that a class $\mathfrak{X}$ of finite groups is a **Schunck class** if it contains every group whenever its primitive factor groups are in $\mathfrak{X}$. Here a finite group is **primitive** if it has a maximal subgroup with trivial core (see [5]).

THEOREM A: *Let $\mathfrak{X}$ be a Schunck class of finite groups. Let the finite group $G = AB = AC = BC$ be the product of two nilpotent subgroups $A$ and $B$ and an $\mathfrak{X}$-subgroup $C$. If for every common prime divisor $p$ of the orders of $A$ and $B$ the cyclic group of order $p$ is an $\mathfrak{X}$-group, then $G$ is a (soluble) $\mathfrak{X}$-group.*

As an application of Theorem A we note the following generalization of the theorem of Amberg and Pennington mentioned above.

COROLLARY 1: *Let $\mathfrak{X}$ be a Schunck class of finite groups such that the product of two normal $\mathfrak{X}$-subgroups is an $\mathfrak{X}$-group. Let the finite group $G = AB$ be the product of two nilpotent subgroups $A$ and $B$. If for every common prime divisor $p$ of the orders of $A$ and $B$ the cyclic group of order $p$ is an $\mathfrak{X}$-group, then the maximal normal $\mathfrak{X}$-subgroup of $G$ is factorized.*

Theorem A becomes false if only one of the two subgroups $A$ and $B$ is nilpotent (and the other is an $\mathfrak{X}$-group), as an example of Peterson shows; see [4], Remark 2.5.12. On the other hand, if $\mathfrak{X}$ is a saturated formation of finite groups, it was shown in [3] and [2] that a finite group $G = AB = AK = BK$ is an $\mathfrak{X}$-group whenever $A$ and $B$ are $\mathfrak{X}$-groups and $K$ is a nilpotent normal subgroup of $G$; see [4], Theorem 6.5.4. The example in section 4 below shows that we cannot expect such a result for arbitrary Schunck classes. Altogether we can say the following.

THEOREM B: *Let $\mathfrak{X}$ be a saturated formation of finite groups or a Schunck class of finite soluble groups. Suppose that the finite group $G = AB = AK = BK$ is the product of two $\mathfrak{X}$-subgroups $A$ and $B$ and a nilpotent normal subgroup $K$ of $G$. Then in both cases $G$ is an $\mathfrak{X}$-group.*

Our last theorem extends part of Theorem B to general group classes. Let $\mathfrak{R}$ be a class of groups closed under the forming of epimorphic images. A $\mathfrak{R}$-**formation** $\mathfrak{F}$ is an epimorphism closed class of $\mathfrak{R}$-groups which is residual with respect to $\mathfrak{R}$, i.e. if $G \in \mathfrak{R}$ and $N_i$ is normal in $G$ such that $G/N_i \in \mathfrak{F}$ for every $i$ in the index set $I$, then $G/\bigcap N_i \in \mathfrak{F}$.

THEOREM C: *Let $\mathfrak{F}$ be a $\mathfrak{R}$-formation which is residual with respect to $\mathfrak{R}$ and satisfies the following condition:*

(∗) *If $X$ is a $\mathfrak{N}$-group with abelian $\mathfrak{F}$-residual $D$, then $D$ has a complement in $X$ and all such complements are conjugate in $X$.*

*Assume further that the $\mathfrak{N}$-group $G = AB = AK = BK$ is the product of two subgroups $A$ and $B$ an a nilpotent normal subgroup $K$ of $G$. If $A$ and $B$ are $\mathfrak{F}$-groups, then also $G$ is an $\mathfrak{F}$-group.*

It is well-known that for a saturated formation of finite soluble groups, condition (∗) of Theorem C holds; see [5], Theorem 5.18, p. 383. This can be generalized to the class of $\mathfrak{U}$-groups introduced by Gardiner, Hartley and Tomkinson in [7] and defined by the following conditions:

(i) Subgroups of $\mathfrak{U}$-groups are $\mathfrak{U}$-groups,

(ii) Every $\mathfrak{U}$-group has a finite (subnormal) series with locally finite-nilpotent factors,

(iii) If $G$ is a $\mathfrak{U}$-group and $\pi$ is a set of primes, then the maximal $\pi$-subgroups of $G$ are conjugate.

Clearly, every $\mathfrak{U}$-group is locally finite-soluble. It is shown in [7] that in the class of $\mathfrak{U}$-groups saturated formations can be defined and condition (∗) of Theorem C can be proved. Therefore the following generalization of a part of Theorem B can be deduced from Theorem C.

Corollary 2: *Let $\mathfrak{F}$ be a saturated $\mathfrak{U}$-formation. Let the $\mathfrak{U}$-group $G = AB = AK = BK$ be the product of three subgroups $A$, $B$ and $K$, where $K$ is nilpotent and normal in $G$. If $A$ and $B$ are $\mathfrak{F}$-groups, then $G$ is an $\mathfrak{F}$-group.*

Y. Sysak [18] has constructed locally finite-soluble groups of the form $G = AB = AK = BK$ where $A$ and $B$ are isomorphic $q$-subgroups for a Mersenne prime $q$ and a normal $p$-subgroup $K$ for the prime $p$. These groups $G$ are not locally nilpotent, although the three subgroups $A$, $B$ and $K$ are locally nilpotent. This indicates limitations in generalizing Theorem C and Corollary 2.

*Notation:* The notation is standard and can for instance be found in [4] and [5]. We note in particular that a group $G$ is an $\mathfrak{X}$-**group** if it belongs to the class of groups $\mathfrak{X}$.

## 2. Primitive products of finite nilpotent groups

A finite group $G$ satisfies $D_\pi$ for the set of primes $\pi$ if every $\pi$-subgroup of $G$ is contained in some Hall $\pi$-subgroup and if any two Hall $\pi$-subgroups of $G$ are conjugate. It is well-known that every finite soluble group satisfies $D_\pi$ for every set of primes $\pi$.

LEMMA 1 (Pennington [15]): *Let the finite group $G = AB$ satisfying $D_\pi$ be the product of two subgroups $A$ and $B$. If $A$ and $B$ are both $\pi$-closed, then $A_\pi B_\pi = B_\pi A_\pi$ is a Hall $\pi$-subgroup of $G$ and $[A_\pi, B_\pi] \subseteq O_\pi(G)$. In particular, if $O_\pi(G) = 1$, then $[A_\pi^G, B_\pi^G] = 1$.*

We note the following lemma on primitive factorized finite soluble groups.

LEMMA 2: *Let the primitive finite soluble group $G = AB$ be the product of two subgroups $A$ and $B$, and let $M$ be a uniquely determined minimal normal subgroup of $G$ such that $M = C_G(M) \neq G$. Then the following holds:*

(a) *$M$ is an elementary abelian $p$-group for some prime $p$ and $M = \operatorname{Fit} G = O_P(G)$,*
(b) *The Fitting subgroup $L/M$ of $G/M$ is a non-trivial $p$-group and $Z(L) = 1$,*
(c) *If $A$ and $B$ are $p$-closed, then $A$ or $B$ is a $p$-group.*

*Proof:* (a) This is clear; see [5], Chapter A, section 15.

(b) Obviously the Fitting subgroup $L/M$ of $G/M$ is a non-trivial $p'$-group. Since $Z(L) \subseteq C_G(M) = M$ and $L$ does not centralize $M$, it follows that $Z(L) = 1$.

(c) If $A$ is not a $p$-group, we have $A_{p'} \neq 1$, and the minimal normal subgroup $M$ of $G$ lies in $A_{p'}^G$. Since $O_{p'}(G) = 1$, by Lemma 1 we have

$$[M, B_{p'}] \subseteq [A_{p'}^G, B_{p'}^G] = 1.$$

Hence

$$B_{p'} \subseteq C_G(M) = M = O_p(G).$$

Thus $B_{p'} = 1$, so that $B$ is a $p$-group. This proves (c). ∎

The following proposition gives some information on primitive finite groups which are factorized by two nilpotent subgroups (see also Gross [8] and Heineken [9]).

PROPOSITION: *Let the primitive group $G = AB$ be the product of two nilpotent subgroups $A$ and $B$ and let $M \neq G$ be a uniquely determined minimal normal subgroup of $G$.*

(i) *$M$ is an elementary abelian $p$-group and $M = C_G(M) = \operatorname{Fit} G = O_p(G)$,*
(ii) *$M = (A \cap M)(B \cap M)$ and $A \cap B = 1$,*

(iii) *A and B are maximal nilpotent subgroups of G,*

(iv) *A or B is a Sylow-p-subgroup of G, the other is a p′-group,*

(v) *If $G = AC = BC$ for some subgroup C of G, then $G = C$.*

*Proof:* (i) The group $G$ is soluble by the theorem of Kegel and Wielandt, so that (i) follows from Lemma 2.

(ii) The Fitting subgroup of $G$ is factorized, so that $M = (A \cap M)(B \cap M)$ and $A \cap B \subseteq M$; see [4], Lemma 2.5.7. In particular $A \cap B$ is a $p$-group. The Fitting subgroup $L/F$ of $G/F$ is a $p'$-group and $Z(L) = 1$ by Lemma 2. The subgroup $A \cap B$ is centralized by $M$ and by a suitable $p$-complement $G_{p'}$ of $G$. Therefore $A \cap B$ is contained in $Z(L) = 1$. This proves (ii).

(iii) Assume that the nilpotent subgroup $A$ is properly contained in a nilpotent subgroup $P$ of $G$. By the modular law $P = A(B \cap P)$. Clearly $G = PB$, so that by (ii) it follows that $P \cap B = 1$. Hence $A = P$. This proves (iii).

(iv) By Lemma 2 one of the two subgroups $A$ or $B$ is a $p$-group. If $A$ is a $p$-group, it is even a Sylow-$p$-subgroup of $G$ by (iii). Since $A \cap B = 1$, it follows that $A$ is a $p'$-subgroup of $G$. This proves (iv).

(v) If $G = AC = BC$ and $(|A|, |B|) = 1$, the order of $A$ and the order of $B$ both divide the order of $C$. Hence by (iv) $|G| = |A||B| = |C|$ and so $G = C$. ∎

The preceding proposition is very useful for the study of finite products of nilpotent groups. As an application of this we give a short proof of the following criterion for a finite product of two nilpotent groups to belong to the saturated formation of metanilpotent groups, i.e. to have Fitting length at most two. Note, however, that a finite product of two nilpotent groups may have arbitrary large Fitting length, since for each positive integer $k$ there exists a group of order $p^m q^n$ and Fitting length $k$ where $p$ and $q$ are primes and $m$ and $n$ are natural numbers.

THEOREM (Maier [13]): *Let the finite group $G = AB$ be the product of two nilpotent subgroups A and B. Then G is metanilpotent if at least one of the following two conditions is satisfied:*

(i) *The p-length $l_p(G) \leq 1$ for every prime p,*

(ii) *Any two subgroups of A commute and any two subgroups of B commute.*

*Proof:* Assume that the theorem is false, and let $G = AB$ be a counterexample of minimal order. Since $A$ and $B$ are nilpotent, $G$ is soluble by the theorem of Kegel and Wielandt. Clearly $G$ is not metanilpotent. Since the metanilpotent groups form a saturated formation, $G$ is a primitive group with uniquely determined minimal normal subgroup $M = C_G(M) = \mathrm{Fit}\, G = O_p(G) \neq G$ which is an elementary abelian $p$-group for some prime $p$.

By the proposition the orders of $A$ and $B$ are relatively prime. If $l_p(G) \leq 1$ it follows that $G/M$ is a $p'$-group. Therefore $M = A$ and so $G = MB$ is metanilpotent. Therefore we may assume that (ii) holds.

Clearly $A \neq B$. By [11] there exists a proper normal subgroup $N$ of $G$ containing $A$ or $B$; see [4], Theorem 7.1.6. If $A \subseteq N$, then $N = A(N \cap B) = AB*$. By induction $N$ is metanilpotent. It is easy to see that $M = \mathrm{Fit}\, N$. Therefore $N/M$ is nilpotent and hence a $p'$-group. Hence $M$ is a Sylow-$p$-subgroup of $N$ and $M = A_p B*_p$ is the product of the $p$-subgroup $A_p$ of $A$ and the $p$-subgroup $B*_p$ of $B*$.

By the proposition one of the two subgroups $A$ or $B$ is a $p$-group, the other a $p'$-group. In particular $A \cap B = 1$. Then $M = A_p = A$ or $M = B*_p$. In the first case $G$ is metanilpotent, so that we may assume that $M = B*_p$.

Then $M = B*_p = B*$ is a maximal normal subgroup of $B$. Since the Frattini subgroup of $G$ is trivial, there exists a complement $U$ of $M$ in $G$. Then $Q = B \cap U$ is a complement of $M$ in $B$. By [13], Lemma 3, $B$ is abelian, contradicting $M = C_G(M)$. The theorem is proved. ∎

## 3. Proof of Theorem A and Corollary 1

*Proof of Theorem A:* Assume that the theorem is false, and let $G$ be a counterexample of minimal order. Then $G$ is not an $\mathfrak{X}$-group, but every proper epimorphic image of $G$ is an $\mathfrak{X}$-group. This implies that $G$ is primitive. If $G$ is nilpotent, it is therefore cyclic of prime order $p$. If $A = B = G$, then $G$ is an $\mathfrak{X}$-group by hypothesis. If $A$ or $B$ is trivial and the other is the whole group $G$, then $G = C$ is an $\mathfrak{X}$-group. Also in the remaining case when $G$ is not nilpotent, $G = C$ is an $\mathfrak{X}$-group by the proposition. This proves Theorem A.

*Proof of Corollary 1:* The product $F$ of all normal $\mathfrak{X}$-subgroups of the finite group $G$ is a normal $\mathfrak{X}$-subgroup of $G$. The factorizer of $F$ in $G$ is

$$X(F) = F(A \cap BF) = F(B \cap AF) = (A \cap BF)(B \cap AF).$$

Hence $X = X(F)$ is an $\mathfrak{X}$-group by Theorem A. Since $A$ and $B$ are nilpotent, the subgroup $X/F$ is subnormal in $G/F$, so that $X$ is subnormal in $G$; see [4], Corollary 6.3.11. Therefore $X$ is contained in $F$ and $F = X(F)$ is factorized.

## 4. Proof of Theorem B and the example

*Proof of Theorem B:* Assume that theorem is false, and let $G = AB = AK = BK$ be a counterexample of minimal order. Then $G$ is not an $\mathfrak{X}$-group, but every proper epimorphic image of $G$ is an $\mathfrak{X}$-group. This implies that $G$ is a primitive group. In particular Frat $G = 1$. Since $K \neq 1$ is nilpotent, it follows that $G$ has exactly one minimal normal subgroup $M$, which is an elementary abelian $p$-group for some prime $p$. Thus

$$M = C_G(M) = K.$$

Evidently $A$ and $B$ are maximal subgroups of $G$. Moreover, since $K$ is contained in every non-trivial normal subgroup of $G$, the images of $A$ and $B$ are $\mathfrak{X}$-maximal in every epimorphic image of $G$. Thus $A$ and $B$ are $\mathfrak{X}$-projectors of $G$, so that in both cases they are conjugate. In the case of a saturated formation $\mathfrak{X}$ this follows from a result of Schmid [17]; in the case of a Schunck class $\mathfrak{X}$ of finite soluble groups it follows from [5], Theorem 3.21, p. 299. Finally this implies that $G = A = B$ is an $\mathfrak{X}$-group. This contradiction proves Theorem B.

*Example:* Let $V(3,2)$ be a 3-dimensional vector space over GF(2) and let $G$ be a semidirect product of $V(3,2)$ with $\mathrm{GL}(3,2)$, the general linear group of $3 \times 3$ matrices over GF(2), under the natural action of $\mathrm{GL}(3,2)$ on $V(3,2)$. The group $\mathrm{GL}(3,2)$ is simple of order 168. Then $G = KA$ where the normal subgroup $K$ of $G$ is elementary abelian of order 8 and $A$ is isomorphic with $\mathrm{GL}(3,2)$. Let $\mathfrak{X}$ be the smallest Schunck class containing the simple group $\mathrm{GL}(3,2)$. It follows that

$$\mathfrak{X} = \{X; X/\mathrm{Frat}X \text{ is isomorphic to } \mathrm{GL}(3,2) \text{ or } 1\},$$

see Erickson [6], p. 1921. Since $A$ is isomorphic to $\mathrm{GL}(3,2)$ it is a (maximal) $\mathfrak{X}$-subgroup of $G$ and $K$ is an abelian minimal normal subgroup of $G$. Furthermore, the first cohomology group $H^1(A,K)$ consists of exactly 2 elements, and hence there is a non-trivial 1-cocycle $f$ as described in Holt and Plesken [10], p. 172, which maps $A$ surjectively onto $K$. The set $B = \{af(a) \colon a \in A\}$ is easily seen to be a complement of $K$ in $G$. Thus $B$ is a (maximal) $\mathfrak{X}$-subgroup of $G$ which is isomorphic to $A$. Finally we see that $G = AB$ by the surjectivity of $f$ as follows.

First observe that $\{af(u); a, u \in A\}$ consists of precisely $168 \times 8$ elements, since $K \cap A = 1$ and as the set $\{f(a) \colon a \in A\}$ has exactly 8 elements. Now for each $a$

in $A$, it follows that $f(a) = a^{-1}(af(a)) \in AB$. Therefore $G = AB = AK = BK$, since the order of $G$ is $168 \times 8$. Clearly $A$ and $B$ are $\mathfrak{X}$-groups, but $G$ is not an $\mathfrak{X}$-group.

## 5. Proof of Theorem C

Assume that Theorem C is false and consider a counterexample $G = AB = AK = BK$ where the nilpotent class $c$ of $K$ is minimal. Assume first that $c = 1$, so that $K$ is abelian.

Since $G/K = AK/K \simeq A/(A \cap K)$ is an $\mathfrak{F}$-group, the $\mathfrak{F}$-residual $D$ of $G$ is contained in $K$. Therefore also $D$ is abelian. By condition $(*)$ $D$ has a complement $Y$ in $G$, so that $K = D \times (Y \cap K)$. The subgroup $T = Y \cap K$ is normal in $AY = G$. Since $G/D$ is an $\mathfrak{F}$-group and $G$ is not an $\mathfrak{F}$-group, it follows that $G/T$ is not an $\mathfrak{F}$-group.

Assume first that $K = D$. Since $K$ is abelian, $A \cap K$ and $B \cap K$ are normal subgroups of $AK = BK = G$, so that also $C = (A \cap K)(B \cap K)$ is normal in $G$. The $\mathfrak{N}$-group $G/C$ has the triple factorization

$$G/C = (AC/C)(BC/C) = (AC/C)(K/C) = (BC/C)(K/C)$$

with $\mathfrak{F}$-subgroups $AC/C \simeq A/(A \cap C)$ and $BC/C \simeq B/(B \cap C)$ and an abelian normal subgroup $K/C$. Then

$$(AC/C) \cap (K/C) = 1 = (BC/C) \cap (K/C).$$

By $(*)$ the complements $AC/C$ and $BC/C$ are conjugate, so that $G/C$ is an $\mathfrak{F}$-group. Hence $D$ is contained in $C$. This implies that $K = (A \cap K)(B \cap K)$. Therefore $G/(A \cap K) \simeq B/(B \cap A \cap K)$ is an $\mathfrak{F}$-group. Hence $K = D \subseteq A \cap K \subseteq A$, and so $G = A$ is an $\mathfrak{F}$-group. This contradiction shows that $D$ must be properly contained in $K$.

Consider next the triply factorized $\mathfrak{N}$-group

$$G/T = (AT/T)(BT/T) = (AT/T)(K/T) = (BT/T)(K/T).$$

We claim that the $\mathfrak{F}$-residual $S/T$ of $G/T$ is equal to $K/T$. Since $(G/T)(K/T) \simeq G/K$ is an $\mathfrak{F}$-group, it follows that $S/T$ is contained in $K/T$, so that $S$ lies in $K$. On the other hand, $G/S \simeq (G/T)/(S/T)$ is an $\mathfrak{F}$-group. Hence $D$ is contained in $S$. In fact, since $T$ lies obviously in $S$, it is clear that $K = D \times T \subseteq S$. Therefore

$K = S$, which proves the claim. Thus we obtain that $G/T$ is an $\mathfrak{F}$-group, a contradiction. Therefore $K$ is not abelian.

Suppose now that $c > 1$. By the minimality of $c$ we have that $G/Z(K)$ is an $\mathfrak{F}$-group. Hence the $\mathfrak{F}$-residual $D$ is contained in $Z(K)$, so that $D$ is abelian. By $(*)$ $D$ has a complement $Y$ in $G$. Since $D$ lies in $Z(K)$, the subgroup $Y \cap K$ is normal in $DY = G$. Thus $K = D \times (Y \cap K)$, and this implies that $K/(Y \cap K)$ is abelian. It follows from the abelian case that $G/(Y \cap K)$ is an $\mathfrak{F}$-group. However, since $G/D$ is an $\mathfrak{F}$-group, also $G \simeq G/(Y \cap K \cap D)$ is an $\mathfrak{F}$-group. This proves Theorem C.

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